Application of Graph Theory on the Advantage of Going First in Hex

Raden Francisco Trianto Bratadiningrat - 13522091[1]
Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
[1]*13522091@itb.ac.id*

***Abstract*— Hex, a two-player strategic board game played on a hexagonal grid, has intrigued mathematicians and game theorists due to its simplicity and depth. Invented by Piet Hein in 1942 and later reinvented by John Nash, Hex has become the base of exploration in various mathematical disciplines. In this paper, we will prove the rule that Hex always has a winner and never ties. We have dived into the advantage of going first in Hex. Although not proven, we have shown that the color going first in Hex is more likely to win.**

***Keywords*—Hex, Hex strategy, Graph Theory, Discrete Mathematics**

I. INTRODUCTION

Hex, an abstract strategy game played on a hexagonal grid, has long captivated the brilliance of mathematicians, game theorists, and aficionados alike. Known for its simplicity yet strategic depth, Hex has become a subject of mathematical exploration across various disciplines. This paper undertakes the advantageous implications of the initial move in Hex, employing the methodologies afforded by graph theory.

Hex unfolded on a board consisting of hexagonal cells with the size 11x11, and it can vary from 4x4 to 19x19. The objective is to construct a path linking opposing sides of the board and making a connection impermeable by the opponent. Unlike other grid-based games, like chess and go, Hex employs a hexagonal structure, connecting each cell with either six, four, or three neighboring cells. This simple difference introduces a richness of strategic possibilities. Each player alternately places their colored hexagons on the board, creating a web of connections that intertwine and clash as the game progresses.

Strategic depth in Hex revolves around the inherent asymmetry created by the starting sides assigned to each player. Consequently, whether the first move bestows a strategic advantage becomes a subject of conjecture and fascination.

In this paper, we delve into the representation of Hex as a graph with vertices and edges. Through the lens of graph theory and induction, we try to prove the advantage of going first in Hex.

II. GRAPH THEORY

Graphs represent discrete objects and the relation between those objects. By definition, a graph $G = (V, E)$ where V is a finite set of vertices $\{v_1, v_2, \ldots, v_n\}$, and E is a subset of V, called edges that connects the vertices $\{e_1, e_2, \ldots, e_n\}$ [1].

A. Simple and Non-simple Graph

On the existence of graph loops and multiple edges, there are two types of graphs: simple graphs and non-simple graphs. A simple graph is a graph that does not contain any graph loops or multiple edges, while a non-simple graph is a graph that has either graph loops or multiple edges.

There are two types of non-simple graphs: multi-graphs and pseudo-graphs. A multi-graph is a graph that contains double or more edges that connect two vertices. A pseudo-graph is a graph with an edge that connects a vertex to itself, making a loop.

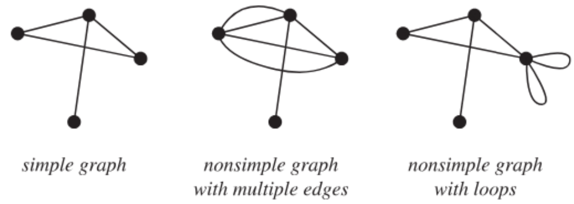

Fig. 1 Types of graphs
(Source: Wolfram Math World https://mathworld.wolfram.com/Graph.html).

Figure 1 shows three graphs. The first graph is a simple graph with four vertices and three edges. The second graph is a non-simple graph with four vertices and five edges with two connections containing two edges, making it a multi-graph. The last graph is a non-simple graph with four vertices and five edges with two looping edges, making it a pseudo-graph.

B. Directed and Undirected Graph

Based on the direction of edges in a graph, there are two types: Undirected Graphs and Directed Graphs. An Undirected Graph is a graph with edges not having a direction, whereas a Directed Graph is a graph in which each edge has a specific direction.

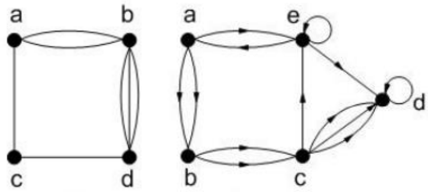

Fig. 2 Undirected and Directed Graph
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

C. Terminologies

1. Adjacent

Two Vertexes are adjacent if both vertices are connected directly. In Figure 3, vertices 1 and 3 are adjacent, while 1 and 4 are not [1].

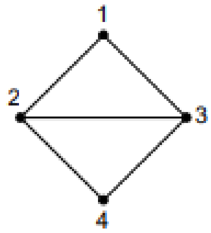

Fig. 3 Adjacent vertex
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

2. Incidence (Connected)

For any edge *e = (vj, vk)*, e is incidence with vertex *vj*, or e is incidence with vertex *vk*. In Figure 4, e1 is incidence with vertexes 1 and 2 [1].

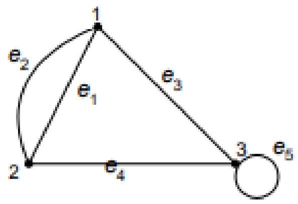

Fig. 4 Incidence of a vertex
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

3. Isolated Vertex

   An isolated vertex is vertex that does not have an edge that is incidence [1].

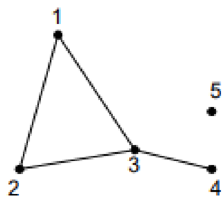

Fig. 5 Isolated Vertex
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

4. Empty Graph

   Graph that has an empty set of edges

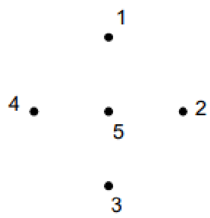

Fig. 6 Empty Graph
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

5. Degree

   The degree of a vertex is the number of edges connected to the vertex [1]. In figure 7, we can see the degree of each vertex.

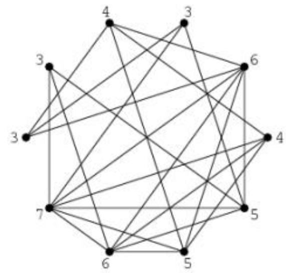

Fig. 7 Degree of vertices
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

6. Path

   A connection of edges starting from a vertex to an end vertex. In Figure 8, example of path: 0-1-2-3-7-9-10.

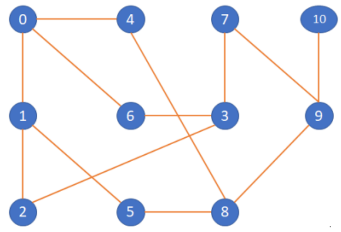

Fig. 8 Graph paths
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

7. Cycle or Circuit

   A path that starts on a vertex and ends on the same vertex. In Figure 8, example of circuit: 0-1-2-3-6-0.

8. Connected

   Two vertices x and y are connected, if there is a path from vertex x to vertex y. In Figure 9, vertices 5 and 7 are connected, although not directly.

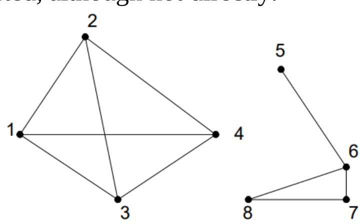

Fig. 9 Graph Connection
(source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf).

III. Combinatorics

Combinatorics is a branch of Mathematics that counts the number of ways in which an object combines without enumerating every structure or combination [2].

In this paper, we will use simple combinatorics to find the quantity of game state possible in the game of Hex.

A. Sum Rule

An object with ***n****-diverse* **or** ***m****-diverse* ways to arrange, with both ways being different, then there are ***(n + m)*** ways to solve the task.

B. Product Rule

An object with *n-diverse* **and** *m-diverse* ways to arrange, with both ways being different, then there are *(n x m)* ways to solve the task.

Inside the game of Hex, a Hex tile has three valid states: empty, taken by player one, or taken by player two. It means that there are three to the power of one (3^1) state possible for a tile. For n number of tiles, there are three to the power of n (3^n) amount of state possible.

IV. Hex

A. Hex rules and basics

Hex, played by two players, on a diamond-shaped board made of hexagonal cells. The board dimensions vary, but the typical size is 11 × 11. Two opposite sides of the board are the same color. In this case, we labeled it using the color black. The remaining two sides can be labeled using the color white. The players alternate turns to place their tile on any unoccupied space on the board to form an unbroken chain of tiles (in their respective colors) linking their two regions [3].

There are many variations of the board size. Typically, in a standard game, it uses the 11x11 size.

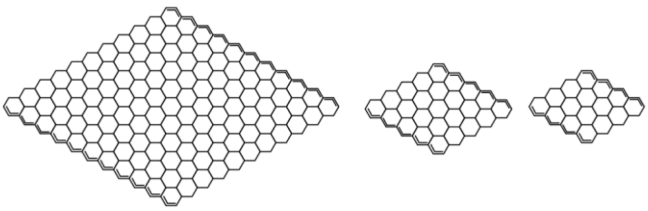

Fig. 10 Variation of board (from the left, 11x11, 5x5, 4x4) (source: writer archive)

The colors black and white often vary and depend. Some variations use the colors red and blue or other combinations of colors.

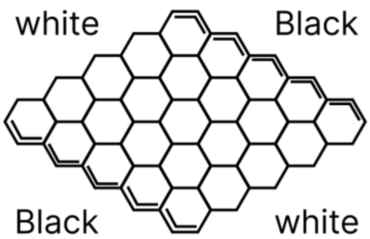

Fig. 11 a 5x5 Hex Board (source: writer archive)

In Figure 11. The white side is the left top and the right bottom of the board, while the black side is the right top and the left bottom. The player with the color white goal is to connect the left top side to the right bottom side of the board. The player with the color black goal is to connect the right top side to the left bottom side of the board. See Figure 12.

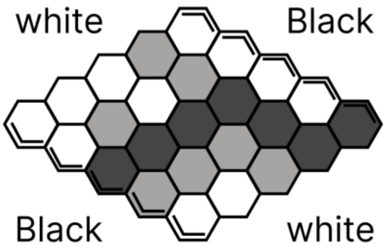

Fig. 12 black wins hex (source: writer archive)

Black won the game. Black successfully connected both sides of the board first, resulting in a win and a loss for white.

Depending on the number of neighboring tiles connected, there are three types of hex tiles: tiles with three neighbors, four neighbors, and six neighbors. See Figure 13.

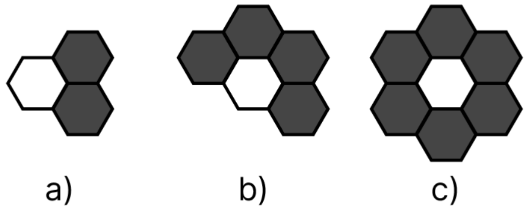

Fig. 13 a) three neighbors, b) four neighbors, c) six neighbors (source: writer archive)

In the rules of Hex, there is a rule called Swap. This rule states that after the first hex tile, the opponent can choose to swap the colors. For example, if the first tile is colored white, placed by Player 1, and the second player, Player 2, chooses to Swap. It then changes Player 1 color to black and Player 2 color to white. After the Swap, the player with the color white will be next to place his tile as the game continues as usual. This rule only applies to the first tile placed. In this paper, we shall ignore the Swap rule because it will only change which player has what color. Not which color that goes first. The color that goes first is the one that placed the first tile, and nothing can change it.

B. Proving the Rule: There is Always a Winner

In the game of Hex, there is always a winner and never a tie. We can prove this by using graph theory and looking into the connections between opposing colors of tile placed on the board.

Suppose a filled board where every Hex tile is not empty, and there is a winner. See Figure 14.

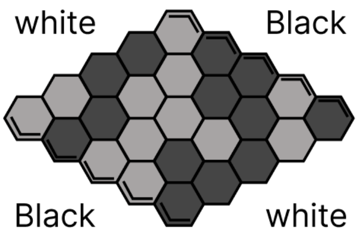

Fig. 14 A filled 5x5 board (source: writer archive)

A hex board is a diamond-shaped board of hex tiles that are hexagon-shaped. We can think about the corner point of a hexagon as the vertex. See Figure 15.

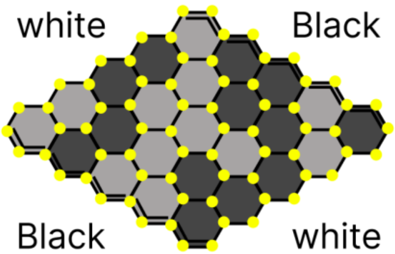

Fig. 15 Vertices in a Hex board
(source: writer archive)

Now connect the edges where a hexagon of color meets a different color, or a hexagon of color touches the opponent's sides of the board. See Figure 16.

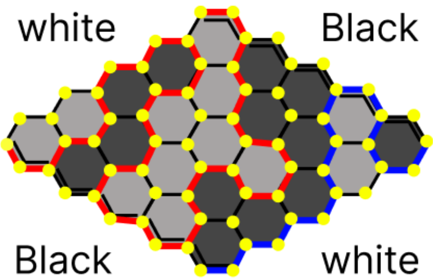

Fig. 16 Graph, with corner points of a hexagon as the vertex.
(source: writer archive)

In Figure 16, There are two distinct graphs filled with connections of edges represented by the red and blue connections. The red connection connects the left corner of the board with the top corner, and the blue connection connects the right corner with the bottom. We can ignore all isolated vertices.

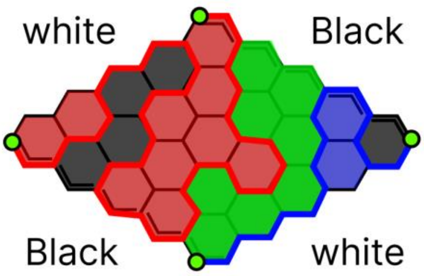

Fig. 17 Hex board with colored areas
(source: writer archive)

In Figure 17, we have removed all the isolated vertices and colored the area according to their connections. The green dots are vertices that become the corners of the board. This vertex of the corners of the board is particularly important. We can see that the two connections connect two green dots or two corners. Because of the connection between these dots, it shows there is a path of the hexagons connecting the side between the separate lines. This path wins the game, as indicated by the green-colored area.

To prove that there are always two connections that connect the four corners, we need to understand the base of each connection. At the base is the junction between three hex tiles. See Figure 18.

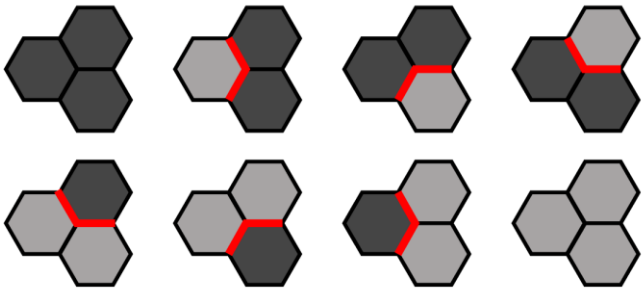

Fig. 18 All junctions possible
(source: writer archive)

In Figure 18, we can see all the possible junctions between two distinct colors, black and white. The red line is the edges differentiating between the two colors. The figure also tells us that there are only two options for the amount connected edge. It can either be two edges or none.

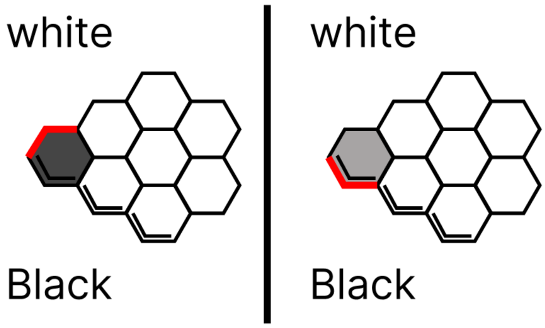

Fig. 19 connection at the corner of the board
(source: writer archive)

In Figure 19, we can also conclude that there are always two graphs of connecting edges at the corner, no matter which corner or what color. We already know that each junction has two or no edges connected, meaning that this graph will grow and cannot stop in the middle or at a junction. Otherwise, there will be a contradiction since a junction has only one connecting edge, which is illegal and invalid. This graph also cannot connect to itself, making a circuit. Otherwise, it also creates a junction with three edges connected, which is illegal and invalid, see Figure 20.

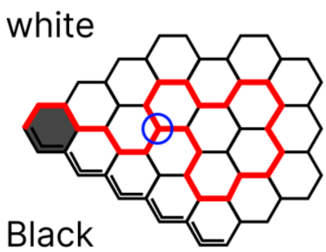

Fig. 20 Illegal junction marked by the blue circle.
(source: writer archive)

Since the graph cannot connect to itself, the only choice is to connect with the other corner, just like what we saw in Figure 15. If the top corner connects to the left corner and the bottom corner connects to the right corner, that means there is a path of hexagon connecting the top right side of the board to the bottom left, so does the opposite. When the top corner connects to the right corner and the bottom corner connects to the left corner, there is a path of hexagon connecting the top left side of the board to the bottom right side. See Figure 21.

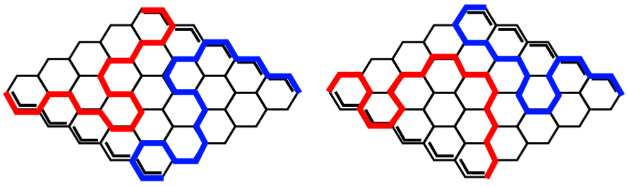

Fig. 21 Two ways of connecting the corners.
(source: writer archive)

Thus, there is proof that there is always a winner in the game of Hex. The argument of this proof has no consideration for the size of the board, meaning it applies to every variant size of the board. We can say that in Hex using any variant size of Hex boards, there is always a winner and never a tie.

C. Strategies and Going First

Since we have proven that there is always a winner, we now can go into strategies of winning and can ignore all possibilities of drawing or strategies that focus on making a tie. This meant that the following strategies discussed, are very important to win the game.

Hex requires complex strategies to win. Most strategies are mathematically intense. In hex, the basic includes [4]:

- Play defensively: defense is also attack.
- Use bridges to make connections between your pieces and simultaneously to block your opponent.
- If you can think of a strong response to your own move, then look for a better one!
- Never give up the game until it is clearly over but abandon areas of the board which are hopeless.

To better understand the strategies of Hex, we can represent the board as a graph. Different from the representation of proving the rule, this graph uses the center of the hexagon as the vertices and the connection between the hexagon as the edges.

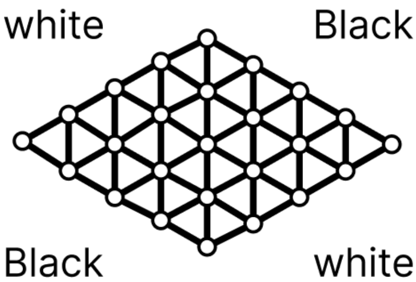

Fig. 22 Representation of Hex with hexagon as vertices and connection as edges
(source: writer archive)

Using this representation, we can better look at the state of the board. To make things clearer, we shall use red as the one going first and blue as the opponent.

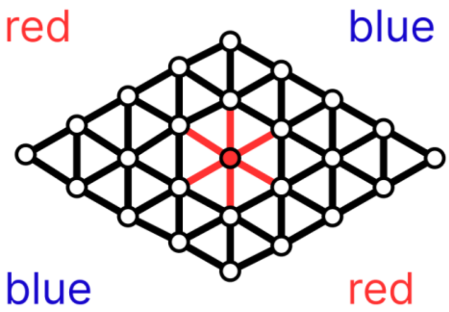

Fig. 23 Red first move.
(source: writer archive)

At the early stage of the game, looking at the potential of moves is particularly important. In Figure 23, where red goes first, we can see that choosing the tile with the largest number of neighbors will open more possibilities to either attack or defend. More possibilities mean more ways to react or win the game.

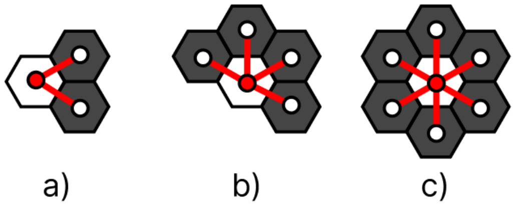

Fig. 24 Connection with a) three neighbors b) four neighbors c) six neighbors
(source: writer's archive)

In going first, to get the highest possibility of winning, the player must choose the tile with the maximum number of connecting tiles.

In Hex, some strategies and principles are essential to winning, some of them include:

A. The two Bridge

The formation consists of two non-adjacent pieces but has two empty neighboring hexes. This arrangement is called a two-bridge [5]. In Figure 23, the green dots are potential connections that connect the two red tiles. If either of the green dots is taken by the opponent, red can simply take the other one.

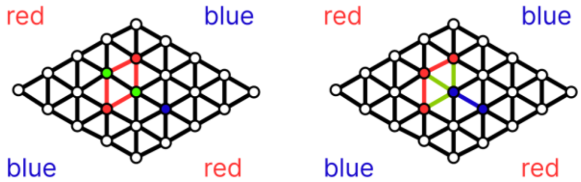

Fig. 25 Two bridge strategy
(source: writer archive)

B. Blocking moves

When you have no pieces in the area, it is usually best to start blocking at a distance. But if you block too close, the opponent can simply connect around the attempted block. [5].

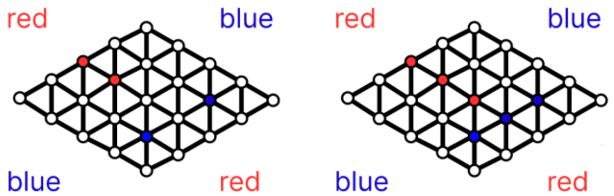

Fig. 26 Blocking moves.
(source: writer archive)

C. Momentum

The player who is dictating the play is said to have the momentum. Alternatively, the momentum is against the player that must respond to the opponent. The player with the momentum usually has the advantage. This advantage is often very decisive. You should try not to hand over the momentum to the opponent unless you have a particularly good reason for doing so. In well-played close matches, the momentum often swings between the two players with each move [5].

D. Multiple threats per move

The player who is dictating the play is said to have the momentum. Alternatively, the momentum is against the player who is being forced to respond to the opponent. The player with the momentum usually has the advantage and this advantage is often decisive. You should generally not hand over the momentum to the opponent unless you have a very good reason for doing so. In well played close matches, the momentum often swings between the two players with each move [5].

E. The center

The central region of the board is strategically the most critical area. From the center, connections can spread out in all directions giving you more flexibility and options than starting from an edge. Furthermore, centrally played pieces are equidistant from both of your edges. The greater the distance apart two pieces are, the harder they are to connect, i.e., their potential link is weaker [5].

Going first in hex has a few advantages. One of them is being the one with the first momentum because you are the one who places the first tile. By correctly using the momentum and always acting so the momentum never falls to the enemy, we can say that the first player will be able to easily win. But the first player must make sure not to get blocked by the enemy. This might be easy on a 4x4 or 5x5 board but on a typical 11x11 board. Blocking can be a lot easier, making it a key to fighting the opponent's momentum and regaining it.

Having the first move also makes it easier to control the center as you are always ahead by one tile against the opponent. This can be seen in this example in Figure 27.

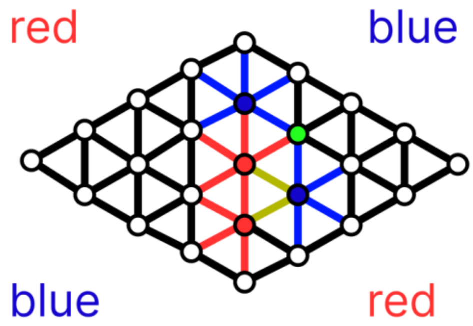

Fig. 27 Controlling the center.
(source: writer archive)

We can see that red is controlling the center because red goes first. Figure 18, red starts the game by taking the very center of the board. This strategy only works in a 5x5 board with extraordinarily little chance to do blocking moves. The green dot is of the tile red can take. Taking the green tile blocks the connection between the two blue tiles while giving more potential links. In an 11x11 board, the same strategy of taking the center can be used, but it can be very dangerous.

Using the two-bridge strategy will be the most important way to keep the momentum and ensure that each hex placed can connect. And since going first means you can make the first bridge to ensure that if the opponent tries to go through, you can easily block it since it guarantees a connection, no matter what your opponent does unless it's a fault by the player going first.

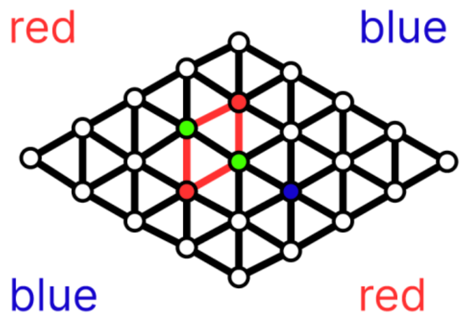

Fig. 28 The two-bridge strategy.
(source: writer archive)

In Figure 19, we can see that the two green dots cause that if blue takes one of the tiles, then red can just take the other one. This formation guarantees that red can always connect their hex tiles. Making it so that going first is an advantage but keep in mind that we are ignoring the swap rule since it just changes which color a player gets, but the color that goes first is the one that gets the advantage in the end.

Connecting all these strategies is the fact that there is always a winner in hex, making all actions either help or ruin your chances to win.

IV. Testing

To assess our theory that going first has an advantage, we implemented a hex game that randomly generates a game position for a 5x5 board. See appendix A for the program and the documentation.

Table 1 Result of Testing using a program that randomly generated a hex game on a 5x5 board.

Samples	First Player	Opponent	Win Percentage of going first
100	53	47	53.000 %
1.000	582	418	58.199 %
10.000	5667	4333	56.670 %
100.000	57475	42525	57.475 %
1.000.000	573724	426276	57.372 %
10.000.000	5731246	4268754	57.312 %
100.000.000	57326873	42673127	57.326 %

Table 1, the result of testing using the randomly generated game. We use a sample size of 100 to 100 million (100.000.000) samples. The result shows that in a randomly generated game, it shows that the first player is more likely to win a game.

In a 5x5 board, there are twenty-five cells with each cell having a state of empty, filled by player one, or filled by the opponent. There are 325 or 827.288.609.443 game states, but since a game cannot be empty to end, we can narrow it down to just two states, filled by player one or by the opponent. We get 225 or 33.554.432 game states, but this amount assumes that all games played are full without a single empty cell, assuming a minimum of 9 hex tile places for a side to win in a 5x5 board of hex. That means that the amount of game state can be:

$$Total\ state\ where\ a\ side\ wins = \sum_{i=9}^{25} 2^{(25-i)}$$

The result is 391.804.103.331 or 391 million states. Because our max sample is 100 million, which is far from 391 million states, even if we assume that all 100 million samples are unique, we are still far from proving that the first player always wins.

Still, since the number of samples assessed is large and yet the winning percentage of going first is still higher on all occasions. We can say that going first in hex is indeed more likely to win the game.

V. Conclusion

Furthermore, the evidence gathered from the implementation of Hex supports the notion that going first provides a distinct advantage. The data from randomly generated game positions consistently shows a higher likelihood of the player making the first move and emerging victory. This aligns with the theoretical claims and reinforces the strategic importance of the initial move in Hex.

While this research successfully establishes the rule that there is always a winner and that going first has advantages, it is important to note that it does not conclusively prove that the first player is always the superior side. The nature of Hex, influenced by various strategies and player decisions, warrants further research and exploration.

In conclusion, this study contributes valuable insights into Hex by combining theoretical foundations with practical testing. The combination of Graph Theory, strategic analysis, and combinatorics enriches our understanding of Hex, making it a captivating subject for both mathematical exploration and strategic gameplay.

VI. Appendix

Appendixes A: documentation of the writer's implementation of Hex and its testing result.

VII. Acknowledgment

The writer would like to thank all IF2120 lecturers, especially Dr. Fariska Zakhralativa Ruskanda, S.T., M.T. as the lecturer in our class IF2120 for Discrete Mathematics, for teaching and supporting students in creating these papers. I have gained a much better understanding of graph theory and its application on Hex. I also would like to thank Dr. Ir. Rinaldi, M.T., who provided students with plentiful resources on Discrete Mathematics on the website.

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Pernyataan

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

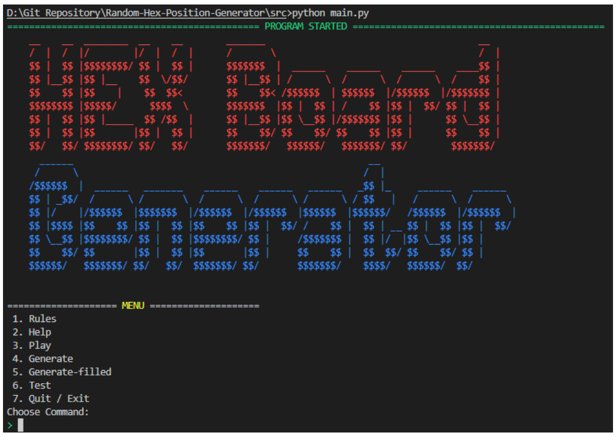
Bandung, 11 Desember 2023

R. Francisco Trianto B. 13522091

APPENDIX A

documentation of the writer's implementation of Hex and its testing result. You can find the code in this link: https://github.com/NoHaitch/Hex-With-Random-Position-Generator

Starting the program:



Generating a random game, in this game blue won

```
================ GENERATING ================
      1   2   3   4   5
    ======================
   \ / \ / \ / \ / \ / \
 A  | . | . | X | O | O |  A
     \ / \ / \ / \ / \ / \
   B  | X | . | O | X | X |  B
       \ / \ / \ / \ / \ / \
     C  | O | X | O | X | O |  C
         \ / \ / \ / \ / \ / \
       D  | X | O | X | O | X |  D
           \ / \ / \ / \ / \ / \
         E  | O | X | O | . | O |  E
             \ / \ / \ / \ / \ / \
              ======================
              1   2   3   4   5
=========== FINISH GENERATING ============
```

Test result using samples 100:

```
How many random games do you want to do?
> 100

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 53
Player 2 wins : 47
Win rasio for player 1 : 53.0%
```

Test result using samples 1.000:

```
How many random games do you want to do?
> 1000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 582
Player 2 wins : 418
Win rasio for player 1 : 58.199999999999996%
```

Test result using samples 10.000:

```
How many random games do you want to do?
> 10000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 5667
Player 2 wins : 4333
Win rasio for player 1 : 56.67%
```

Test result using samples 100.000:

```
How many random games do you want to do?
> 100000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 57475
Player 2 wins : 42525
Win rasio for player 1 : 57.475%
```

Test result using samples 1.000.000:

```
How many random games do you want to do?
> 1000000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 573724
Player 2 wins : 426276
Win rasio for player 1 : 57.3724%
```

Test result using samples 10.000.000:

```
How many random games do you want to do?
> 10000000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 5731246
Player 2 wins : 4268754
Win rasio for player 1 : 57.31246%
```

Test result using samples: 100.000.000

```
> 100000000

==================== TESTING ====================

==================== FINISHED ====================
RESULT :
Player 1 wins : 57326873
Player 2 wins : 42673127
Win ratio for player 1 : 57.326873%
```